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Some solutions of a supersymmetric non-polynomial oscillator—a comparison between the SWKB and WKB methods

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Abstract. It is shown that the non-polynomial oscillator represented by $V(x)=x^2+\lambda x^2/(1+gx^2)$ can be given a supersymmetric form if the parameter λ satisfies a certain constraint. In this case we find an exact analytic expression for the ground state. Excited states corresponding to this potential are computed for several values of g (and λ) using the supersymmetric WKB method, the ordinary WKB method and also numerical integration. The results are compared.

1. Introduction

Supersymmetric quantum mechanics (SUSYQM) was introduced by Witten [1] as a laboratory to test supersymmetry breaking. However, at present SUSYQM is not just a laboratory to study to SUSY breaking but is widely studied in various other contexts [2]. Very recently Comtet *et al* [3] formulated a supersymmetric version of the WKB method (SWKB). Later this method was applied in the cases of some exactly solvable potentials [4] (for which the exact bound-state spectrum is reproduced) as well as some non-exactly solvable models [5].

In the SWKB approach, first of all the potential has to be supersymmetrised and this naturally sets constraints on the parameters of the theory (thereby truncating the range of values of the parameters). However, even with this built-in limitation the method would be considered powerful if it produced reasonably good results (compared to other methods, e.g., the ordinary WKB method) for such a restricted range of parameters. In this paper our purpose will be to examine this issue.

Here we shall show that when the parameter λ is restrained by a particular relation the interaction [6] $V(x)=x^2+\lambda x^2/(1+gx^2)$ becomes supersymmetric. In this case an exact analytic expression for the ground state can be found out almost trivially. Although the non-polynomial interaction has been studied comprehensively by perturbative as well as numerical methods [7-10], no WKB evaluation has been carried out so far. In the present paper we shall compute the excited states (as well as the ground state in some cases) of the supersymmetrised non-polynomial oscillator by the SWKB method, the ordinary WKB method and also numerical integration techniques. The computation of energy eigenvalues will be carried out in two steps: in the first phase the computation will be done for $E_-^0=0$, while in the second phase we shall perform the computation with $E_-^0\neq 0$ (it should be mentioned here that no SWKB computation has been carried out so far with non-vanishing ground-state energy). This will allow us to make a comparison between the performance of the SWKB method

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and the WKB method when compared against accurate values obtained by numerical integration. It may be mentioned here that the energy values we have obtained by numerical integration are usually different from those obtained by Bessis and Bessis [11] and Lai and Lin [12] due to our choice of SUSY values of λ; however, when our choice of (λ, g) values agrees with the standard choice [11, 12], the energy values agree identically.

The organisation of the paper is as follows: in § 2 we present an overview of SUSYQM [13] and derive the constraint due to which the non-polynomial interaction becomes supersymmetric; § 3 contains a description of the SWKB method and computation of the energy values by different methods; and, finally, § 4 is devoted to a discussion of the results obtained.

2. SUSYQM in one dimension and the non-polynomial interaction

We recall that in one dimension the Hamiltonian of a SUSYQM system consists of the following pair of Hamiltonians [13]:

$$H_s = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{1}$$

$$H_\pm = -\mathrm{d}^2/\mathrm{d}x^2 + V_\pm(x) \tag{2}$$

$$V_\pm(x) = W^2(x) \pm W'(x) \tag{3}$$

where $W(x)$ is called the superpotential. The ground-state (of zero energy) wavefunctions corresponding to $H_\pm$ are given by

$$\varphi_\pm^0(x) \sim \exp\left(\pm \int^x W(t)\, \mathrm{d}t \right) \tag{4}$$

and the ground state is normalisable if $\varphi_\pm^0(x) \to 0$ as $x \to \pm\infty$.

Next, to show how the non-polynomial interaction represented by

$$V(x) = x^2 + \frac{\lambda x^2}{(1+gx^2)} = x^2 + \frac{\lambda}{g} - \frac{\lambda}{g}\frac{1}{(1+gx^2)} \tag{5}$$

fits into the SUSY framework, we choose the following superpotential:

$$W(x) = \mu x - 2gx/(1+gx^2). \tag{6}$$

The scalar potentials $V_\pm(x)$ corresponding to (6) are given by

$$V_-(x) = \mu x^2 + \frac{2g+4\mu}{(1+gx^2)} - 5\mu \tag{7}$$

$$V_+(x) = \mu^2 x^2 - \frac{2g-4\mu}{(1+gx^2)} + \frac{8g^2x^2}{(1+gx^2)^2} - 3\mu. \tag{8}$$

It is clear that we have to identify the non-polynomial interaction with $V_-(x)$ and if (5) is identical with (7) we have

$$-\lambda/g = (2g+4\mu) \tag{9}$$

$$\mu = \pm 1 \tag{10}$$

and the relation between the energy eigenvalues is given by

$$E_- + 5\mu = E_{NP} - \lambda/g. \tag{11}$$

Hence if we know E_- then E_{NP} becomes known through

$$E_{NP} = E_- + 5\mu + \lambda/g. \tag{11'}$$

Note that the ground-state wavefunctions are given by (obtained from (4))

$$\varphi_+^0(x) \sim (1+gx^2)^{-1} \exp(\tfrac{1}{2}\mu x^2) \tag{12}$$

$$\varphi_-^0(x) \sim (1+gx^2) \exp(-\tfrac{1}{2}\mu x^2). \tag{13}$$

From (12) and (13) it follows that if $\mu = +1$ then $\varphi_-^0(x)$ is normalisable (with $E_-^0 = 0$) while if $\mu = -1$ then $\varphi_+^0(x)$ is normalisable (with $E_+^0 = 0$).

3. Spectrum of the SUSY non-polynomial oscillator

In this section we shall find an exact analytic expression for the ground state whenever λ satisfies (9) and $\mu = +1$. For this case the excited states are computed using the SWKB, the WKB and numerical integration methods. For $\mu = -1$, $E_- \neq 0$ and all the states, including the ground state, will be found by the above methods.

Let us consider $\mu = +1$. In this case we have

$$\varphi_-^0(x) \sim (1+gx^2) \exp(-\tfrac{1}{2}x^2) \tag{14}$$

$$E_{NP}^0 = \lambda/g + 5 \tag{15}$$

where

$$-\lambda/g = (2g+4). \tag{16}$$

To test the credibility of the above results let us check them against the values of the parameters used by other authors. To this end we take

$$g = 0.1 \qquad \lambda = -0.42. \tag{17}$$

Then from (14) and (15) we get

$$\varphi_-^0(x) \sim (1+\tfrac{1}{10}x^2) \exp(-\tfrac{1}{2}x^2) \tag{18}$$

$$E_{NP}^0 = 0.8 \tag{19}$$

and this agrees with the result of [14].

Before moving over to the computation of the excited states let us briefly describe the SWKB method [3-5]. First note that the standard WKB quantisation rule is (we consider the bosonic sector):

$$\int \mathrm{d}x[E_-^n - (W^2 - \hbar W')]^{1/2} = \pi(n+\tfrac{1}{2})\hbar. \tag{20}$$

Expanding the LHS of (20) in powers of $\hbar$ we obtain

$$\int_a^b \mathrm{d}x(E_-^n - W^2)^{1/2} + \tfrac{1}{2}\hbar \int_a^b \mathrm{d}x \frac{W'}{(E_-^n - W^2)^{1/2}} + \ldots = \pi(n+\tfrac{1}{2})\hbar \tag{21}$$

where a and b are the turning points defined by

$$W^2(a) = W^2(b) = E_-^n. \tag{22}$$

Therefore the SWKB quantisation condition to leading order in $\hbar$ is

$$\int_a^b \mathrm{d}x(E_-^n - W^2)^{1/2} = \pi n\hbar, \qquad n = 0, 1, 2, \ldots. \tag{23}$$

Preceeding similarly, the SWKB quantisation condition for the fermionic sector can be found to be [3]

$$\int_a^b \mathrm{d}x(E_+^n - W^2)^{1/2} = \pi(n+1)\hbar. \tag{24}$$

From (23) and (24) it is easily seen that level degeneracy holds, i.e.

$$E_-^{n+1} = E_+^n. \tag{25}$$

For the non-polynomial interaction some of the relevant expressions are

$$\int_a^b \mathrm{d}x\left(E_-^n - \mu^2x^2 + \frac{4\mu g x^2}{(1+gx^2)} - \frac{4g^2x^2}{(1+gx^2)^2}\right)^{1/2} = \pi n\hbar \tag{26}$$

$$\int_c^d \mathrm{d}x\left(E_-^n - \mu^2x^2 - \frac{2g+4\mu}{(1+gx^2)} + 5\mu\right)^{1/2} = \pi(n+\tfrac{1}{2})\hbar \tag{27}$$

where the former is the SWKB result and the latter is the ordinary WKB result.

The integrals on the LHS of (26) and (27) can be evaluated by an iteration procedure (we skip the algebraic details and the results are summarised in tables 1 and 2).

Table 1. Energy eigenvalues together with percentage error for $\mu = +1$ and $E_-^0 = 0$. Values obtained by numerical integration for $g = 0.1$ and $\lambda = -0.42$ agree identically with values obtained by Fack *et al* [14].

g	λ	n	WKB	Percentage error	SWKB	Percentage error	Numerical integration
0.05	−0.205	0	0.8962	0.422	0.9000	0	0.9000
		1	2.7122	0.084	2.7151	0.022	2.7145
		2	4.5531	0.032	4.5554	0.017	4.5546
		3	6.4134	0.015	6.4153	0.014	6.4144
		4	8.2892	0.008	8.2908	0.010	8.2899
		5	10.1778	0.005	10.1791	0.007	10.1783
0.1	−0.42	0	0.7848	1.9	0.8000	0	0.8000
		1	2.4510	0.191	2.4598	0.166	2.4557
		2	4.1959	0.047	4.2016	0.088	4.1979
		3	5.9905	0.015	5.9946	0.053	5.9914
		4	7.8197	0.005	7.8227	0.033	7.8201
		5	9.6744	0.001	9.6767	0.022	9.6745
0.15	−0.645	0	0.6666	4.77	0.7000	0	0.7000
		1	2.2166	0.130	2.2313	0.531	2.2195
		2	3.9032	0.033	3.9117	0.184	3.9045
		3	5.6646	0.003	5.6701	0.093	5.6648
		4	7.4738	0.022	7.4776	0.053	7.4736
		5	9.3163	0.003	9.3191	0.033	9.3160
0.2	−0.88	0	0.5435	9.41	0.6000	0	0.6000
		1	2.0060	0.194	2.0257	1.17	2.0021
		2	3.6558	0.024	3.6660	0.025	3.6567
		3	5.3988	0.014	5.4050	0.129	5.3980
		4	7.1984	0.008	7.2025	0.065	7.1978
		5	9.0361	0.006	9.0390	0.038	9.0335

Table 2. Energy eigenvalues with $\mu = -1$ and $E_-^0 \neq 0$. Values obtained by numerical integration for $\lambda = 2.0$, $g = 1.0$ agree identically with values obtained by Bessis and Bessis [11].

g	λ	n	WKB	Percentage error	SWKB	Percentage error	Numerical integration
0.1	0.38	0	1.1649	0.682	1.1551	0.164	1.1570
		1	3.4446	0.127	3.4376	0.075	3.4402
		2	5.6723	0.044	5.6672	0.045	5.6698
		3	7.8614	0.017	7.8576	0.030	7.8600
		4	10.0211	0.006	10.0182	0.021	10.0204
		5	12.1581	0.003	12.1557	0.016	12.1577
0.2	0.72	0	1.2825	1.648	1.2533	0.665	1.2617
		1	3.7109	0.202	3.6941	0.025	3.7034
		2	6.0172	0.041	6.0068	0.131	6.0147
		3	8.2468	0.008	8.2399	0.075	8.2461
		4	10.4257	0.001	10.4210	0.046	10.4258
		5	12.5699	0.002	12.5664	0.030	12.5702
0.3	1.02	0	1.3699	2.545	1.3179	1.347	1.3359
		1	3.8817	0.201	3.8563	0.454	3.8739
		2	6.2125	0.020	6.1984	0.206	6.2112
		3	8.4434	0.004	8.4348	0.106	8.4438
		4	10.6148	0.008	10.6091	0.062	10.6157
		5	12.7481	0.007	12.7441	0.038	12.7490
0.4	1.28	0	1.4350	3.319	1.3593	2.13	1.3889
		1	3.9902	0.150	3.9578	0.662	3.9842
		2	6.3200	0.006	6.3035	0.267	6.3204
		3	8.5386	0.015	8.5288	0.129	8.5399
		4	10.6957	0.011	10.6893	0.071	10.6969
		5	12.8152	0.008	12.8108	0.042	12.8163
0.5	1.5	0	1.4825	3.984	1.3830	2.995	1.4257
		1	4.0541	0.071	4.0159	0.871	4.0512
		2	6.3700	0.023	6.3515	0.627	6.3715
		3	8.5612	0.023	8.5606	0.146	8.5732
		4	10.7126	0.014	10.7058	0.077	10.7141
		5	12.8188	0.009	12.8140	0.046	12.8200
1.0	2.0	0	1.5309	5.776	1.3281	8.236	1.4473
		1	3.9820	0.410	3.9265	1.798	3.9984
		2	6.1778	0.011	6.1524	0.422	6.1785
		3	8.2914	0.042	8.2764	0.223	8.2949
		4	10.3677	0.007	10.3576	0.105	10.3685
		5	12.4235	0.010	12.4162	0.069	12.4248

4. Discussion of the results

In this paper we have computed the energy eigenvalues of the non-polynomial oscillator when the coupling constants λ and g satisfy a supersymmetric constraint. The computation has been done by the supersymmetric WKB method, the WKB method and numerical integration.

The computation of the eigenvalues has been carried out in two phases. In the first phase we have considered $\mu = +1$ resulting in vanishing ground-state energy

($E_-^0 = 0$). In this case the SWKB method gives the exact value of the ground-state energy of the non-polynomial oscillator. On the other hand, the standard WKB method gives a reasonable approximation to the ground-state energy. However, it can be seen from table 1 that, as n increases, the WKB method starts giving better results.

The second part of the computation deals with the more interesting case $\mu = -1$. In this case the ground-state energy is non-vanishing, i.e. $E_-^0 \neq 0$ and neither the WKB method nor the SWKB method give the exact value of the ground-state energy. However, even in this case the ground-state energy values calculated via the SWKB method come nearer to the exact values than those calculated via the WKB method. But with increasing n the previous trend continues, i.e. WKB values are in excellent agreement with the exact values while the SWKB values are in reasonable agreement.

Therefore, we conclude that for $n > 1$ the WKB method is superior to the SWKB method while for $n = 0, 1$ the reverse is true. It is, however, possible that in the $E_-^0 \neq 0$ case the SWKB values may improve if $O(\hbar^2)$ corrections are taken into account (although the same correction could also be applied to the WKB case).

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